SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 16

MATH 241

13.1.2

Proof. Suppose $u(x, y) = X(x)Y(y)$, then the equation becomes

$$X'Y + 3XY' = 0$$

$$\frac{X'}{X} = -3\frac{Y'}{Y}$$

This is a number that depends on neither x nor y, call it λ, then $X' = \lambda X$, $Y' = -\frac{1}{3}Y$. Therefore $X = C_1e^{\lambda x}$, $Y = C_2e^{-\frac{1}{3}\lambda y}$, and $u(x, y) = Ce^{\lambda x - \frac{1}{3}y}$. □

Remark If you tried the method I told you to solve a first order linear PDE, you need to first substitute $x = t + s, y = 3t$, therefore $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y}$, so the equation becomes $\frac{\partial u}{\partial t} = 0$, this means $u(s, t) = g(s)$ is a function of s itself. Then you write s in terms of x, y: $s = x - \frac{1}{3}y$, so $u(s, t) = g(x - \frac{1}{3}y)$, it can be any function of $x - \frac{1}{3}y$, for example, $x - \frac{1}{3}y$, $(x - \frac{1}{3}y)^2$, $\cosh(x - \frac{1}{3}y)$. The solutions we get by the method of separation of variable are $Ce^{\lambda x - \frac{1}{3}y}$, and in particular, they are indeed a function of $x - \frac{1}{3}y$. This example shows that the solutions with variables separated are not all the solutions in general. That is the limit of this method.

13.1.8

Proof. Suppose $u(x, y) = X(x)Y(y)$, then the equation becomes

$$yX'Y' + XY = 0$$

$$\frac{X'}{X} = -\frac{Y}{yY'}$$

This is a number that depends on neither x nor y, call it λ, then $X' = \lambda X$, $Y' = -\frac{1}{\lambda y}Y$.

For the first equation, you get $X = C_1e^{\lambda x}$, for the second equation, you get a first order linear ODE $Y' + \frac{1}{\lambda y}Y = 0$, the integral factor is $e^{\int \frac{1}{\lambda y}dy} = e^{\frac{1}{\lambda}\ln y} = y^{\frac{1}{\lambda}}$, so multiply $y^{\frac{1}{\lambda}}$ you get $y^{\frac{1}{\lambda}}Y' + \frac{1}{\lambda}y^{\frac{1}{\lambda}-1}Y = 0$, $(y^{\frac{1}{\lambda}}Y)' = 0$, this means $y^{\frac{1}{\lambda}}Y = C_2$, $Y = \frac{C_2}{y^{\frac{1}{\lambda}}}$. So $u(x, y) = C\frac{e^{\lambda x}}{y^{\frac{1}{\lambda}}}$, or written as $Ce^{\lambda x - \frac{1}{\lambda}\ln y}$.

□

13.1.19

Proof. $A = 1, B = 6, C = 9, \Delta = B^2 - 4AC = 0$, so it is parabolic. □

13.1.28

Proof. By direct checking.

$$\frac{\partial u}{\partial r} = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 \alpha r^{\alpha-1} - c_4 \alpha r^{-\alpha-1})$$

$$\frac{\partial^2 u}{\partial r^2} = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 \alpha(\alpha-1) r^{\alpha-2} + c_4 \alpha(\alpha+1) r^{-\alpha-2})$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 \alpha r^{\alpha-2} - c_4 \alpha r^{-\alpha-2})$$

Therefore

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = (c_1 \cos \alpha\theta + c_2 \sin \alpha\theta)(c_3 \alpha^2 r^{\alpha-2} + c_4 \alpha^2 r^{-\alpha-2}) = \frac{\alpha^2}{r^2} u$$

$$\frac{\partial u}{\partial \theta} = (-c_1 \alpha \sin \alpha\theta + c_2 \alpha \cos \alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha})$$

$$\frac{\partial^2 u}{\partial \theta^2} = (-c_1 \alpha^2 \cos \alpha\theta - c_2 \alpha^2 \sin \alpha\theta)(c_3 r^{\alpha} + c_4 r^{-\alpha}) = -\alpha^2 u$$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{\alpha^2}{r^2} u$$

so

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

□

13.2.2

Proof. Left end's temperature is held at u_0 means $u|_{x=0} = u_0$, right end's temperature is held at u_1 means $u|_{x=L} = u_1$, initial temperature is $u|_{t=0} = 0$. So the boundary value problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

$$u(0,t) = u_0, u(L,t) = u_1$$

$$u(x,0) = 0$$

□

13.2.4

Proof. The heat transfer at the left end implies that there is a constant h such that $\frac{\partial u}{\partial x}|_{x=0} = h(u|_{x=0} - 20)$. Right end is insulated means there is no heat transfer there, hence $\frac{\partial u}{\partial x}|_{x=L} = 0$. Therefore the boundary value problem is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial u}{\partial x}(0,t) = h(u(0,t) - 20), \frac{\partial u}{\partial x}(L,t) = 0$$
$$u(x,0) = f(x)$$

□

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Proof. (a)

Let $u = X(x)T(t)$, so the equation becomes $XT' = X'T$, therefore $\frac{X'}{X} = \frac{T'}{T}$, this constant depends on neither x nor t, call it λ, then $X' = \lambda X$, $T' = \lambda T$, so $X = C_1 e^{\lambda x}, T = C_2 e^{\lambda t}$, therefore $u(x,t) = Ce^{\lambda x + \lambda t}$.

(b)

This is a nonhomogeneous equation, so we need to find a special solution first. Since the right hand side is a constant, so we can say $u = u(t)$ only depends on t, and we simplify the equation into $\frac{\partial u}{\partial t} = 3$, so we have a special solution $u_0 = 3t$. The associated homogeneous PDE is just the PDE in (a), then we can have a lot of solutions to the nonhomogeneous PDE $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + 3$ as u_0 plus the solutions to (a):

$$u(x,t) = 3t + Ce^{\lambda x + \lambda t}$$

For different λ's, we can have linearly independent answers, and the answer is definitely not unique. For example, let $C = 1, \lambda = 1$ and $C = 1.\lambda = 2$ we can have two solutions $3t + e^{x+y}$ and $3t + e^{2x+2y}$, and they are linearly independent, i.e., if there exists two real numbers c_1, c_2 such that $c_1(3t + e^{x+y}) + c_2(3t + e^{2x+2y}) = 0$, then the only possibility is $c_1 = c_2 = 0$. That is apparent. □